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THE STABILITY OF A NON-AUTONOMOUS FUNCTIONAL-DIFFERENTIAL EQUATION RELATIVE TO PART OF THE VARIABLES†

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The asymptotic stability and instability of the trivial solution of a functional-differential equation of delay type relative to part of the variables are investigated using limit equations and a Lyapunov function whose derivative is sign-definite. The theorems thus obtained are used to solve the problem of stabilizing mechanical control systems with delayed feedback. As examples, solutions of problems of the uniaxial and triaxial stabilization of the rotational motion of a rigid body with a delay in the control system are presented. © 1999 Elsevier Science Ltd. All rights reserved.

1. BASIC DEFINITIONS. LIMIT EQUATIONS.

Suppose R^m and R^p are linear real spaces of m- and p-vectors with norms $|\mathbf{y}|$ and $|\mathbf{z}|$, R^n is the linear real space of n-vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_p) = (\mathbf{y}, \mathbf{z})$ with norm $|\mathbf{x}| = |\mathbf{y}| + |\mathbf{z}|$, $n = m + p$, $h > 0$ is a real number and $\mathbf{C}^{(n)}$ is the Banach space of continuous functions $\boldsymbol{\varphi}$: $[-h, 0] \to R^n$ with norm $\|\boldsymbol{\varphi} = \sup(|\boldsymbol{\varphi}(s)|, -h \leqslant s \leqslant 0)$, $C_H^{(m)} = \{\boldsymbol{\varphi}_{(y)} \in C^{(m)} : \|\boldsymbol{\varphi}_{(y)}\| < H < +\infty\}$, $\overline{C}_l^{(m)} = \{\boldsymbol{\varphi}_{(y)} \in C^{(m)} : \|\boldsymbol{\varphi}_{(y)}\| \leqslant l\}$ (and similarly for $\overline{C}_r^{(p)}$). Put $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_{(y)}, \boldsymbol{\varphi}_{(z)})$. For a continuous function $\mathbf{x} :]-\infty, +\infty[\to R^n$ and every $t \in R$, the function $\mathbf{x}_t \in C^{(n)}$ is defined by the equality $\mathbf{x}_t(s) = \mathbf{x}(t + s)$ for $h \leqslant s \leqslant 0$.

Consider the following functional-differential equation with finite delay

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t), \quad \mathbf{f}(t, 0) = 0 \tag{1.1}$$

where $\mathbf{f} : R^+ \times \Lambda \to R^n$, $\Lambda = C_H^{(m)} \times C^{(p)}$ is a continuous mapping satisfying the $\mathbf{z}$-continuability condition for solutions of Eq. (1.1), that is: each solution of Eq. (1.1) $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi}) = \mathbf{x}_\alpha(\alpha, \boldsymbol{\varphi}) = \boldsymbol{\varphi}$ is defined for all $t \geqslant \alpha$ such that $|\mathbf{y}(t, \alpha, \boldsymbol{\varphi})| \leqslant H_1 < H$. This condition means that none of the coordinates $z_j(t, \alpha, \varphi)$ will depart to infinity in a finite time [1].

Let us assume that the right-hand side of (1.1) also satisfies the following assumptions.

Assumption 1.1. For every pair r, l, $0 < r < H$, $l > 0$, an $M = M(r, l)$ exists such that for $(t, \boldsymbol{\varphi}) \in R^+ \times \overline{C}_r \times \overline{C}_l$

$$|\mathbf{f}(t, \boldsymbol{\varphi})| \leqslant M \tag{1.2}$$

With this assumption, we can prove the following lemma [2, 3].

Lemma 1.1. Suppose Assumption 1.1 is satisfied and let $\mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$ be a solution of Eq. (1.1), defined and bounded for all $t \geqslant \alpha - h$. Then the family of functions $\{\mathbf{x}_t(\alpha, \boldsymbol{\varphi}) : t \geqslant \alpha\}$ is precompact.

Assumption 1.2. The function $\mathbf{f}(t, \boldsymbol{\varphi})$ satisfies a Lipschitz condition as a function of φ in any compact set $K \subset \Lambda$, that is, an $l = l(K)$ exists such that, for any $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in K$

$$|\mathbf{f}(t, \boldsymbol{\varphi}_2) - \mathbf{f}(t, \boldsymbol{\varphi}_1)| \leqslant l \, \|\boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_1\| \tag{1.3}$$

Under this condition, a solution of Eq. (1.1) exists for any initial point $(\alpha, \boldsymbol{\varphi}) \in R^+ \times \Lambda \mathbf{x} = \mathbf{x}(t, \alpha, \boldsymbol{\varphi})$, and it is unique and continuous with respect to the initial data [4].

Assumption 1.3. The function $\mathbf{f}(t, \boldsymbol{\varphi})$ is uniformly continuous over any set $R^+ \times K$, where $K \subset \Lambda$ is an arbitrary compact subset of Λ, so that $\forall K \subset \Lambda$, $\forall \varepsilon > 0$ $\exists m = m(K)$, $\exists \delta = \delta(\varepsilon, K) > 0$ such that

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$\forall\ (t_1, \varphi_1), (t_2, \varphi_2) \in R^+ \times K, |t_2 - t_1| \leqslant \delta, \|\varphi_2 - \varphi_1\|\ \delta$, the following relation holds

$$|\mathbf{f}(t_2, \varphi_2) - \mathbf{f}(t_1, \varphi_1)| < \varepsilon$$

Under these conditions, Eq. (1.1) is precompact in some space F of continuous functions $\mathbf{f}$: $R^+ \times \Gamma \to R^n$, where Γ is some subset of Λ that contains $\{\mathbf{x}_t(\alpha, \varphi), \varphi \in \Lambda, t \geqslant \alpha + h\}$ for every solution $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ of (1.1) [3].

Definition 1.1. A function $\mathbf{f}^* : R^+ \times \Gamma \to R^n$ is said to be a limit function for $\mathbf{f}$ if a sequence $t_n \to +\infty$ exists such that $\{\mathbf{f}(t + t_n, \varphi)\}$ is uniformly convergent to $\mathbf{f}^*(t, \varphi)$ in F. The equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t) \tag{1.4}$$

is called a limit equation for (1.1).

The relationship between the solutions of Eqs (1.1) and (1.4) is established by the following theorem [2, 3].

Theorem 1.1. Let $\mathbf{f}^* : R^+ \times \Gamma \to R^n$ be a limit function for $\mathbf{f}$ in F relative to a sequence $t_n \to +\infty$, and let $\{\alpha_n \in R^+\}$ and $\{\varphi_n \in \Gamma\}$ be sequences such that $\alpha_n \to \alpha \in R^+$, $\varphi_n \in \Gamma$ as $n \to \infty$. Then, if $\mathbf{x} = \mathbf{x}(t, t_n + \alpha_n, \varphi_n)$ is a solution of Eq. (1.1) and $\mathbf{x}^*(t, \alpha, \varphi)$ is a solution of the equation $\dot{\mathbf{x}}(t) = f^*(t, \mathbf{x}_t)$ defined for $t \in [\alpha - h, \beta[$, the sequence of functions $\mathbf{x}(t_n + t, t_n + \alpha_n, \varphi_n)$ converges to $x^*(t, \alpha, \varphi)$ uniformly in $t \in [\alpha - h, \gamma]$ for every $\gamma < \beta$.

2. ASYMPTOTIC STABILITY AND INSTABILITY THEOREMS RELATIVE TO PART OF THE VARIABLES, ON THE ASSUMPTION THAT THE SOLUTIONS ARE BOUNDED AS FUNCTIONS OF THE NON-CONTROLLABLE COORDINATES

We will investigate the problem of the stability of Eq. (1.1) relative to part of the variables $(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$, using limit equations and limit functions.

Let $V(t, \varphi) : R^+ \times \Lambda \to R$ be some functional defined and jointly continuous with respect to all its arguments [3]. Let $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ be some solution of Eq. (1.1) defined for all $t \geqslant \alpha - h$. Then, putting $V(t) = V(t, \mathbf{x}_t(\alpha, \varphi))$, we can define the upper right-hand derivative

$$\dot{V}(\alpha, \varphi) = \lim_{h \to 0^+} \sup \frac{1}{h}(V(\alpha + h) - V(\alpha))$$

Let us assume that the derivative $\dot{V}$ satisfies the following estimate

$$\dot{V}(t, \varphi) \leqslant -W(t, \varphi) \leqslant 0, \quad \forall (t, \varphi) \in R \times \Lambda$$

where the continuous function $W = W(t, \varphi)$ is bounded and uniformly continuous on every set $R^+ \times K$, where K is a compact subset of Λ.

Definition 2.1. Let $t_n \to +\infty$ be some sequence. For each $t \in R$ and $c \in R$, we define a set $V_\infty^{-1}(t, c) \subset \Lambda$ as follows: the point $\varphi \in V_\infty^{-1}(t, c)$ if a sequence $\{\varphi_n \in \Gamma, \varphi_n \to \varphi\}$ exists such that $\lim_{n\to+\infty} V(t + t_n, \varphi_n) = c$.

As in the case of $\mathbf{f}(t, \varphi)$, subject to the appropriate condition for $W(t, \varphi)$, the family of translates $\{W^\tau(t, \varphi), \tau \in R^+\}$ is precompact in a certain functional space of continuous functionals $F = \{G : R \times \Gamma \to R\}$ with metrizable compact open topology.

Definition 2.2. A functional $W^* \in F_G$ is called a limit functional for W if a sequence $t_n \to +\infty$ exists such that $\{W^{(n)}(t, \varphi) = W(t_n + t, \varphi)\}$ converges to $W^*(t, \varphi)$ in F_G. When that is the case, the set $V_\infty^{-1}(t, c)$ defined by the same sequence $t_n \to +\infty$ is defined as that corresponding to W^*.

Let $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ be a solution of Eq. (1.1) defined for all $t \geqslant \alpha - h$, $\mathbf{x}_t^{(n)}(\alpha, \varphi) = \mathbf{x}(t_n + s, \alpha, \varphi)$ $(-h \leqslant s \leqslant 0)$. The positive limit set $\Omega^+(\mathbf{x}_t(\alpha, \varphi))$ in the space C_H is the set $\Omega^+ = \{\varphi^* \in C_H: \exists t_n \to +\infty, n \to \infty, \mathbf{x}_t^{(n)}(\alpha, \varphi) \to \varphi^* \text{ as } n \to \infty\}$.

The following results of the localization $\Omega^+(\mathbf{x}_t(\alpha, \varphi))$ exist, easily deducible from [2, 3].

Theorem 2.1. Suppose that

1. $V(t, \varphi) : R^+ \times \Lambda \to R$ is a continuous functional, having a lower bound in every compact set $K \subset \Lambda$, that is, $V(t, \varphi) \geqslant m(K)\ \forall (t, \varphi) \in R^+ \times K$, and for its derivative we have

$$\dot{V}(t,\varphi) \leqslant -W(t,\varphi) \leqslant 0, \quad \forall (t,\varphi) \in R^+ \times \Lambda$$

2. $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ is a solution of Eq. (1.1) such that $|\mathbf{y}(t, \alpha, \varphi)| \leqslant \tau < H$, $|\mathbf{z}(t, \alpha\ \varphi)| \leqslant \tau_1 < +\infty$ for all $t \geqslant \alpha - h$.

Then $c = c_0 \geqslant m$ exists such that, for every limit point $\varphi^* \in \Omega^+(\mathbf{x}_t(\alpha, \varphi))$, a limit pair $(\mathbf{f}^*, W^*)$ with $V_\infty^{-1}(t, c)$ and a solution $\mathbf{x}^*(t, 0, \varphi^*)$ of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$ exist such that $\{\mathbf{x}_t^*(0, \varphi^*) : t \in R\} \subset \Omega^+(\mathbf{x}_t(\alpha, \varphi))$ and $\{\mathbf{x}_t^*(0, \varphi^*) : t \in R\} \subset \{V_\infty^{-1}(t, c) : c = c_0 = \text{const}\} \cap \{W^*(t, \varphi) = 0\}$.

Corollary 2.1. If, under the conditions of Theorem 2.1, it is also assumed that for every $c_0 > c_1$ a certain limit pair $(\mathbf{f}^*, W^*)$ with $V_\infty^{-1}(t, c)$ exists such that the set $\{V_\infty^{-1}(t, c)$: $c = c_0 = \text{const} > c_1\} \cap \{W^*(t, \varphi) = 0\}$ does not contain solutions of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$, then, in addition to the assertion of Theorem 2.1, it is also true that

$$\lim_{t \to +\infty} V(t, \mathbf{x}_t(\alpha, \varphi)) = c_0 = \text{const} \leqslant c_1.$$

These results enable us to prove the following theorems, in which ω: $R^+ \to R^+$ is a Hahn-type function [1].

Theorem 2.2. Suppose that:

1. the solutions of Eq. (1.1) in some neighbourhood N of the point $\mathbf{x} = 0$ are bounded as functions of $\mathbf{z}$;
2. a continuous functional $V : R^+ \times \Lambda \to R^+$ exists such that

$$V(t,0) \equiv 0, \quad V(t,\varphi) \geqslant \omega_1(|\varphi_{(y)}(0)|), \quad \dot{V}(t,\varphi) \leqslant -W(t,\varphi) \leqslant 0$$

for all $(t, \varphi) \in R^+ \times \Lambda$;

3. for every limit pair $(\mathbf{f}^*, W^*)$ with set $V_\infty^{-1}(t, c)$, the set $\{V_\infty^{-1}(t, c) : c = \text{const} \geqslant 0\} \cap \{W^*(t, \varphi) = 0\}$ does not contain solutions of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$ other than the solutions $\mathbf{x} = \mathbf{x}(t) = (\mathbf{y}(t), \mathbf{z}(t))$ such that $\mathbf{y} = 0$.

Then the solution $\mathbf{x} = 0$ of Eq. (1.1) is asymptotically y-stable.

Proof. Condition 2 of the theorem implies that the trivial solution of Eq. (1.1) is y-stable [5]. Consequently, for every number $H_1 < H$ a $\delta = \delta(H_1\ \alpha) > 0$ exists such that if $\|\varphi\| \leqslant \delta$, then $|\mathbf{y}|(t, \alpha, \varphi)| \leqslant H_1$, $\forall t \geqslant \alpha$. Hence, by Condition 1 of the theorem, it follows that every solution $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ of Eq. (1.1) in the neighbourhood $N \cap \{\|\mathbf{x}\| \leqslant \delta\}$ will be such that

$$|\mathbf{y}(t,\alpha,\varphi)| \leqslant H_1 < H, \quad |\mathbf{z}(t,\alpha,\varphi)| \leqslant l < +\infty, \quad \forall t \geqslant \alpha$$

Then, applying Theorem 2.1, we deduce from Conditions 2 and 3 that $\lim_{t\to+\infty}\mathbf{y}(t, \alpha, \varphi) = 0$. The theorem is proved.

Using Corollary 2.1, we prove the following.

Theorem 2.3. Suppose that:

1. Conditions 1 and 2 of Theorem 2.2 are satisfied;
2. a limit pair $(\mathbf{f}^*, W^*)$ with set $V_\infty^{-1}(t, c)$ exists such that, for every $c_0 > 0$, the set $\{V_\infty^{-1}(t, c) : c = c_0 = \text{const} > 0\} \cap \{W^*(t, \varphi) = 0\}$ does not contain solutions of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$.

Then the solution $\mathbf{x} = 0$ of Eq. (1.1) is y-stable uniformly in φ.

Proof. As in the proof of Theorem 2.2, we can deduce that for every solution $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ of Eq. (1.1) defined for all $t \leqslant \alpha$ in the neighbourhood $N \cap \{\|\mathbf{x}\| \leqslant \delta\}$, it is true that $|\mathbf{y}(t, \alpha, \varphi)| \leqslant r < H$, $|\mathbf{z}(t, \alpha, \varphi)| \leqslant l < +\infty$, $\forall t \geqslant \alpha$.

It now follows from Condition 2 of the theorem, according to Corollary 2.1, that along every solution $\mathbf{x}(t, \alpha, \varphi)$ in the domain $N \cap \{\|\mathbf{x}\| \leqslant \delta\}$ we have: $V(t, \mathbf{x}_t(\alpha, \varphi))$ decreases monotonically and tends to zero as $t \to +\infty$.

Suppose the domain $\Lambda_1 \subset \Lambda$ is contained in the stability domain of $\mathbf{x} = 0$ for an arbitrarily given $\alpha \geqslant 0$. Let $\mu > 0$ be an arbitrarily small number. It can be shown, on the basis of Assumption 1.1, that the set $K(\alpha, \alpha + h) = \{\mathbf{x}_{\alpha+h}(\alpha, \varphi) : \varphi \in \Lambda_1\}$ is compact. The property $V(t, \mathbf{x}_t(\alpha, \varphi)) \to 0$, $t \to +\infty$, the continuity of V and the continuous dependence of the solutions on the initial data imply the existence of a number $T = T(\varepsilon, \alpha + h) > 0$ such that

$$V(\alpha+h+T,\ x_{\alpha+h+T}(\alpha+h,\varphi)) < \omega_1(\varepsilon)$$

for all $\varphi \in K(\alpha, \alpha + h)$. Consequently, it follows from the definition of the set $K(\alpha, \alpha + h)$ and Condition 1 of the theorem that, for all $t \geqslant \alpha + h + T$, $\varphi \in \Lambda$, we have $V(t, \mathbf{x}_t(\alpha, \varphi)) < \omega_1(\varepsilon)$. Thus $y_t(\alpha, \varphi) \mid < \varepsilon$, $\forall t \geqslant \alpha$. The theorem is proved.

Theorem 2.4. Suppose that:

1. the solutions of Eq. (1.1) in some neighbourhood N of the point $\mathbf{x} = 0$ are stable relative to $\mathbf{z}$;
2. a continuous functional $V: R^+ \times \Lambda \to [-m, +\infty)$, having a lower band, exists which, in any small neighbourhood of $\mathbf{x} = 0$, takes negative values such that

$$V(t,0) \equiv 0,\quad \dot{V}(t,\varphi) \leqslant -W(t,\varphi) \leqslant 0$$

for all $t \in R^+$, $\varphi \in \Lambda$;

3. for every $c_0 < 0$ a limit pair $(\mathbf{f}^*, W^*)$ with set $V_\infty^{-1}(t, c)$ exists such that the set $\{V_\infty^{-1}(t, c)\colon c = c_0\} \cap \{W^*(t, \varphi) = 0\}$ does not contain solutions of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$.

Then the solution $\mathbf{x} = 0$ of Eq. (1.1) is y-stable.

Proof. Let $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ be a solution of Eq. (1.1) with initial point (α, φ), $\varphi \in N$, $V(\alpha, \varphi) < 0$. We will show that the solution $\mathbf{x}(t, \alpha, \varphi)$ is such that, for any H_1, $0 < H_1 < H$, a $t^* > \alpha$ exists for which $|\mathbf{y}(t^*, \alpha, \varphi)| = H_1$.

Suppose the contrary: for the given solution, $|\mathbf{y}(t, \alpha, \varphi)| < H_1$ for all $t \geqslant \alpha$. Then, by Condition 1 of the theorem, $|\mathbf{z}(t, \alpha, \varphi)| \leqslant l < +\infty$. By Condition 2, a number $c_0 < 0$ exists such that $\lim_{t\to\infty} V(t, \mathbf{x}_t(\alpha, \varphi)) = c_0 < 0$. By Theorem 2.1, a solution $\mathbf{x}^*(t) \subset \Omega^+(\mathbf{x}_t(\alpha, \varphi))$ of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$ exists a solution such that

$$\mathbf{x}^*(t) \subset \{V_\infty^{-1}(t,c) : c = c_0\} \cap \{W^*(t,\varphi) = 0\},\quad \forall t \in R$$

This contradicts Condition 2, proving the theorem.

Theorem 2.5. Suppose that

1. the solutions of Eq. (1.1) in some neighbourhood N of the point $\mathbf{x} = 0$ are uniformly bounded with respect to $\mathbf{z}$;
2. a continuous functional $V: R^+ \times \Lambda \to R^+$ exists, bounded and uniformly continuous in every set $R^+ \times K$ ($K \subset \Lambda$ a compact set), such that

$$\omega_1(|\varphi_{(y)}(0)|) \leqslant V(t,\varphi) \leqslant \omega_2(\|\varphi\|),\quad V(t,0) \equiv 0$$

$$\dot{V}(t,\varphi) \leqslant -W(t,\varphi) \leqslant 0,\quad (t,\varphi) \in R^+ \times \Lambda$$

3. every limit triple $(\mathbf{f}^*, V^*, W^*)$ is such that the set $\{V^*(t, \varphi) = c > 0\} \cap \{W^*(t, \varphi) = 0\}$ does not contain solutions of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$.

Then the solution $\mathbf{x} = 0$ of Eq. (1.1) is uniformly asymptotically y-stable.

Proof. It follows from Condition 2 of the theorem that the trivial solution of Eq. (1.1) is uniformly y-stable [4]. Under those conditions, every solution $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ of Eq. (1.1) in the domain $\Lambda_0 = \{\|\varphi\| \leqslant H_0 = \omega_2^{-1}(\omega_1(H_1)), H_1 < H\}$ is bounded as a function of y, that is $|\mathbf{y}(t, \alpha\ \varphi)| \leqslant H_1 < H$ for all $t \geqslant \alpha$. It follows from Condition 1 of the theorem that the solutions of Eq. (1.1) in the neighbourhood $N_0 = N \cap \Lambda_0$ of the point $\mathbf{x} = 0$ are such that

$$|\mathbf{y}(t,\alpha,\varphi)| \leqslant H_1 < H, |\mathbf{z}(t,\alpha,\varphi)| \leqslant l < +\infty, \forall t \geqslant \alpha$$

As in Theorem 2.3, one deduces that, along every solution $\mathbf{x} = \mathbf{x}(t, \alpha, \varphi)$ in N_0, the functional $V(t, \mathbf{x}_t(\alpha, \varphi))$ decreases monotonically, tending to zero as $t \to +\infty$.

We will show that $V(t, \mathbf{x}_t(\alpha, \varphi))$ as $t \to +\infty$ uniformly with respect to $(\alpha, \varphi) \in R^+ \times N_0$. This property, as follows from Condition 2 of the theorem, will be sufficient to complete the proof.

Suppose the contrary, namely: $\varepsilon_0 > 0$ exists such that, for an arbitrary sequence $t_k \to +\infty$, there is sequence $(\alpha_k, \varphi_k) \in R^+ \times N_0$ for which

$$V(\alpha_k + t_k,\ \mathbf{x}_{\alpha_k + t_k}(\alpha_k, \varphi_k)) \geqslant \varepsilon_0$$

Under these conditions, obviously, the following inequality is true for $t \in [\alpha_k, \alpha_k + t_k]$

$$V(t, \mathbf{x}_t(\alpha_k, \varphi_k)) \geqslant \varepsilon_0 \tag{2.1}$$

Define $\beta_k = \alpha_k + t_k/2$. We may assume, without loss of generality, that the sequences $\{\mathbf{f}^{(k)}(t, \varphi) = \mathbf{f}(\beta_k + t, \varphi)\}$, $\{V^{(k)}(t, \varphi)\}$, $\{W^{(k)}(t, \varphi)\}$ converge to $\mathbf{f}^*(t, \varphi)$, $V^*(t, \varphi)$, $W^*(t, \varphi)$, respectively. By Theorem 1.1, the sequence $\mathbf{x}(\beta_k + t, \beta_k, \psi_k)$, $\psi_k = \mathbf{x}_{\beta_k}(\alpha_k, \varphi_k)$ converges uniformly in $t \in [0, T]$ to a solution $\mathbf{x}^* = \mathbf{x}^*(t, 0, \psi)$ of the equation $\dot{\mathbf{x}}(t) = \mathbf{f}^*(t, \mathbf{x}_t)$.

By (2.1), $V^*(t, \varphi)$ satisfies the estimate

$$V^*(t, \mathbf{x}_t^*(t, 0, \psi)) \geqslant \varepsilon_0 > 0 \tag{2.2}$$

for all $t \geqslant 0$, $\psi \in \overline{C}_{H_1}^{(m)} \times \overline{C}_l^{(p)}$.

On the other hand, it follows from Condition 2 of the theorem that

$$V(t+\beta_k, \mathbf{x}_{t+\beta_k}(\beta_k, \psi_k)) - V(\beta_k, \mathbf{x}_{\beta_k}(\beta_k, \psi_k)) \leqslant -\int_{\beta_k}^{t+\beta_k} W(s, \mathbf{x}_s(\beta_k, \psi_k))ds =$$

$$= -\int_0^t W(s+\beta_k, \mathbf{x}_{s+\beta_k}(\beta_k, \psi_k))ds \leqslant 0,\quad t \geqslant 0$$

Letting $k \to +\infty$ in the last relationship, we obtain

$$V^*(t, \mathbf{x}_t^*(0, \psi)) - V^*(0, \psi) \leqslant -\int_0^t W^*(s, \mathbf{x}_s^*(0, \psi))ds \leqslant 0$$

Hence it follows that the function $V^*(t, \varphi)$ decreases monotonically along the solution $\mathbf{x}^* = \mathbf{x}^*(t, 0, \psi)$ of the equation $\dot{\mathbf{x}} = \mathbf{f}^*(t, \mathbf{x}_t)$ at the same time satisfying inequality (2.1).

It follows from Condition 3 of the theorem that every limit triple for $(\mathbf{f}^*, V^*, W^*)$, say $(\mathbf{f}^{**}, V^{**}, W^{**})$, is such that the set $\{V^{**}(t, \varphi) = c = \text{const} > 0\} \cap \{W^{**}(t, \varphi) = 0\}$ does not contain solutions of the equation $\dot{x} = f^{**}(t, x_t)$. Consequently, $V^*(t, \mathbf{x}_t^*(0, \psi))$ decreases monotonically, tending to zero as $t \to +\infty$. This contradicts inequality (2.2), which proves the theorem.

The results of this section develop and generalize the results of [1–11].

3. STABILIZATION OF MECHANICAL CONTROL SYSTEMS

Consider a mechanical control system obeying holonomic stationary constraints and described by n generalized coordinates $q_1, \ldots, q_n$, which are subject to dissipative forces in addition to control forces $u = (u_1, u_2, \ldots, u_n)$, so that the motion of the system is described by the equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = u_i - \sum_{j=1}^n f_{ij}\dot{q}_j(t)\quad \left(T = \frac{1}{2}\sum_{i=1}^n a_{ij}(q)\dot{q}_i\dot{q}_j\right) \tag{3.1}$$

where T is the kinetic energy of the system and $\|f\|$ is a positive-definite matrix.

We will consider the problem of stabilizing the equilibrium position $\dot{\mathbf{q}} = \mathbf{q} = 0$ of system (3.1) with respect to the coordinates, assuming the presence of delayed feedback. We will show that this problem is solved by a control u of the form

$$u_i = -\frac{\partial \Pi}{\partial q_i}(q_1(t - r_{i1}(t)), \ldots, q_n(t - r_{in}(t))) \tag{3.2}$$

where $\Pi = \Pi(\mathbf{q})$ is a positive-definite function such that

$$\Pi(0) = 0,\quad \sum_{i=1}^n \left(\frac{\partial \Pi}{\partial q_i}\right)^2 \geqslant \omega(\|\mathbf{q}\|)$$

the functions $r_{ij}(t)$ are bounded, $0 < r_{ij} \leqslant h$, and uniformly continuous for $t \in R^+$.

A limit system for (3.1) has the analogous form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}=-\frac{\partial \Pi}{\partial q_i}(q_1(t-r_{i1}^*(t)),\dots,q_n(t-r_{in}^*(t))-\sum_{j=1}^{n} f_{ij}\dot{q}_j(t) \quad (3.3)$$

Let us assume that $\|f\|$ is a positive-definite matrix. Let μ be the least eigenvalue of the matrix $\|f\|$, and let

$$c_0=\max_{i,j}\left|\frac{\partial^2\Pi}{\partial q_i \partial q_j}\right|<\frac{\mu}{hn}$$

We take the following Lyapunov functional

$$V(t,\dot{q}_t,q_t)=T(q_1(t),\dots,q_n(t))+\Pi(q_1(t),\dots,q_n(t))+\int_{-h}^{0}\int_{s}^{0}\sum_{i,j=1}^{n} f_{ij}\dot{q}_j(t+u)\dot{q}_i(t+u)duds$$

This functional will obviously satisfy the conditions

$$\omega_1(|\mathbf{q}(t)|)+\omega_2(|\dot{\mathbf{q}}(t)|)\leqslant V\leqslant \omega_1(\|\mathbf{q}_t\|)+\omega_2(\|\dot{\mathbf{q}}_t\|)$$

For the derivative $\dot{V}$ we obtain the estimate

$$\dot{V}(t,\dot{\mathbf{q}}_t,\mathbf{q}_t)\leqslant -\int_{-h}^{0}\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\mu}{h}-c_0 n\right)\left(\dot{q}_i^2(t)+\dot{q}_i^2(t+s)\right)ds\leqslant 0 \quad (3.4)$$

We put the functional $W(t, \dot{\mathbf{q}}_t)$ equal to the quantity between the inequality signs in (3.4). Then the set $\{\mathbf{W}^*(t, \dot{\mathbf{q}}_t) = 0\} \equiv \{\dot{\mathbf{q}}(t) \equiv 0\}$.

Substituting $\dot{\mathbf{q}} = 0$ into Eq. (3.3), we deduce that the set $\{W^*(t, \dot{\mathbf{q}}_t) = 0\}$ does not contain solutions of Eq. (3.3) other than $\dot{\mathbf{q}} = \mathbf{q} = 0$.

Using Theorem 2.5, we see that the control (3.2) stabilizes the equilibrium position $\dot{\mathbf{q}} = \mathbf{q} = 0$, ensuring uniform asymptotic stability.

In a similar way, it can be shown that if $q_{m+1}, \dots, q_n$ are angular coordinates (mod 2π), then the control (3.2), where $\Pi = \Pi(\mathbf{q})$ is a positive-definite function of $q_1, \dots, q_m$ and

$$\Pi(0)=0\sum_{i=1}^{n}\left(\frac{\partial \Pi}{\partial q_i}\right)^2\geqslant \omega\left(\sum_{i=1}^{m} q_i^2\right)$$

stabilizes the equilibrium position $\dot{\mathbf{q}} = \mathbf{q} = 0$ relative to $\dot{q}_1, \dots, \dot{q}_n, q_1, \dots, q_m$.

These results develop those obtained in [9, 12, 13].

4. EXAMPLES

Synthesis of a control torque ensuring asymptotic stability of a given triaxial orientation of a rigid body, with variable moments of inertia, rotating about a fixed point

Let $OXYZ$ be a fixed reference system and let $Oxyz$ be a reference system rigidly attached to the body, where the axes Ox, Oy and Oz maintain fixed directions in the body and are chosen from the problem of the orientation of $Oxyz$ relative to $OXYZ$. Rotational motion of the body about its centre of mass may be described by the Euler dynamical equations

$$(\dot{\mathbf{I}\omega})+\omega\times\mathbf{I}\omega=\mathbf{M} \quad (4.1)$$

where $\omega = (\omega_x, \omega_y, \omega_z)^T$ is the angular velocity of the body in projections onto the Ox, Oy and Oz axes; $\mathbf{I}(t)$ is the principal inertia tensor in the $Oxyz$ axes, which is assumed to be positive-definite and bounded by the matrix $\mathbf{M} = (M_x, M_y, M_z)^T$ of torques generated by the action of the external forces and displacements of the moving masses of the body.

As kinematic equations, we take the equations in Rodrigues–Hamilton parameters [14]

$$2\dot{\lambda}_0=-\lambda_1\omega_1-\lambda_2\omega_2-\lambda_3\omega_3 \quad (4.2)$$

$$2\dot{\lambda}_1=\lambda_0\omega_1+\lambda_2\omega_3-\lambda_3\omega_2 \ (1\ 2\ 3)$$

Coincidence of the bases $OXYZ$ and $Oxyz$ corresponds to the quaternions $L = (1, 0, 0, 0)$ and $L = (-1, 0, 0, 0)$.

We will solve the triaxial stabilization problem in the following formulation: it is required to find a torque **M**, generated by a control system with delay, which ensures uniform asymptotic stability of the equilibrium position

$$\omega = 0, \quad L = (1, 0, 0, 0) \tag{4.3}$$

We will show that the solution of this problem is obtained by defining the torque **M** as

$$\mathbf{M} = -\mathbf{R}(t)\omega - \alpha\bar{\lambda}(t - r(t)), \quad \bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^T, \quad r(t) \leqslant h, h > 0 \tag{4.4}$$

where $\alpha > 0$ is an arbitrary number and $\mathbf{R}(t)$ is a bounded matrix such that the matrix $(2\mathbf{R}(t) - \dot{\mathbf{I}}(t))$ is positive-definite.

Equations (4.1) may then be rewritten as

$$(\dot{\mathbf{I}\omega}) + \omega \times \mathbf{I}\omega = -\mathbf{R}(t)\omega - \alpha(\bar{\lambda}(t) - \int_{-r(t)}^{0} \dot{\bar{\lambda}}(s)ds). \tag{4.5}$$

If these equations are solved for $\dot{\omega}$, the limit equations will be

$$\dot{\omega} = \{\omega^T \mathbf{A}^*(t)\omega\} + \omega^T \mathbf{B}^*(t) - \alpha \mathbf{C}^*(t)(\bar{\lambda}(t) - \int_{-r^*(t)}^{0} \dot{\bar{\lambda}}(s)ds) \tag{4.6}$$

where $\{\omega^T \mathbf{A}^*(t)\omega\}$ are quadratic forms in $\omega = (\omega_x, \omega_y, \omega_z)^T$ and $\mathbf{B}^*(t)$, $\mathbf{C}^*(t)$ are matrices such that $\det(\mathbf{C}^*(t)) \geqslant c = \text{const} > 0$.

Let a denote the least eigenvalue of the matrix $(2\mathbf{R}(t) - \dot{\mathbf{I}}(t))$. Let $a > 3h\alpha$. Then, for some c_0, we have $a = hc_0 + 3h\alpha$.

We choose a Lyapunov functional in the form

$$V = \omega(t)^T \mathbf{I}(t)\omega(t) + 2\alpha((1 - \lambda_0(t))^2 + \lambda_1^2(t) + \lambda_2^2(t) + \lambda_3^2(t)) +$$
$$+ \int_{-h}^{0}\int_{s}^{0} \frac{3}{2} a(\omega_1^2(t+u) + \omega_2^2(t+u) + \omega_3^2(t+u))duds$$

This functional is positive-definite, has an infinitely small upper limit at the point (4.3) and, by virtue of Eq. (4.5) has a derivative

$$\dot{V} \leqslant -\alpha h \omega^T(t)\omega(t) \leqslant 0$$

The set $\{\omega_x, \omega_y, \omega_z = 0\}$ contains only those solutions of Eqs (4.2) and (4.6) for which $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_0 = +1$ or $\lambda_0 = -1$. By Theorem 2.5, the equilibrium position (4.3) is uniformly asymptotically stable.

The problem of the uniaxial orientation of a rigid body by control torques

Consider a rigid body with a fixed point O. Let $OXYZ$ be an inertial reference system and let $Oxyz$ be a reference system rigidly attached to the body; ψ is the angle of precession, θ is the angle of nutation and φ is the angle of proper rotation. The motion of the body may be described in terms of the Euler angles by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = u_\psi, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = u_\theta, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = u_\varphi \tag{4.7}$$

$$T = \frac{1}{2}(A(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)^2 + B(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)^2 + C(\dot{\varphi} + \dot{\psi}\cos\theta)^2)$$

We will define the control $u = (u_\psi, u_\theta, u_\varphi)^T$ in such a way as to guarantee stabilization of the Ox axis along the OZ axis, that is, stabilization of the set of motions $\{\dot{\varphi} = \dot{\theta} = 0, \dot{\psi} = \text{const}, \varphi = \theta = \pi/2\}$. We will show that the problem is solved by a control of the form

$$u_\psi = 0 \tag{4.8}$$
$$u_\theta = -k_{11}\dot{\theta} - k_{12}\dot{\varphi} - a\sin\varphi(t - r_1(t))\cos(\theta(t - r_2(t)))$$
$$u_\varphi = -k_{21}\dot{\theta} - k_{22}\dot{\varphi} - a\cos\varphi(t - r_1(t))\sin(\theta(t - r_2(t)))$$

where $\| k_{ij} \|$ is a positive-definite matrix, μ is the least eigenvalue of this matrix and $r_i(t)$ are uniformly continuous functions for $t \in R^+$, with $0 < r_i(t) \leqslant h = \text{const}$, $0 < a < \mu/2h$.

The limit equations for (4.7) have a similar form, with right-hand sides that are limiting values for (4.8). Consider the Lyapunov functional

$$V = T + \alpha(1 - \sin\varphi\sin\theta) + \frac{1}{2h}\int_{-h}^{0}\int_{s}^{0}(k_{11}\dot\theta^2(t+u) + 2k_{12}\dot\varphi(t+u)\dot\theta(t+u) + k_{22}\dot\varphi^2(t+u))duds$$

For the derivative of this function, we find that

$$\dot V \leqslant -W \leqslant 0$$

$$W = \int_{-h}^{0}\left(\frac{\mu}{2h} - a\right)(\dot\theta^2(t) + \dot\theta^2(t+s) + \dot\varphi(t)^2 + \dot\varphi^2(t+s))ds$$

The only solution of the limit equation (4.8) in the set $\{W^* = 0\} = \{\dot\varphi = \dot\theta = 0\}$ is

$$\dot\theta = \dot\varphi = 0,\ \ \varphi = \theta = \frac{\pi}{2},\ \ \dot\psi = \text{const}.$$

By Theorem 2.5, we obtain stabilization of the Ox axis along the OZ axis, or asymptotic stability of the equilibrium position

$$\dot\psi = \dot\theta = \dot\varphi = 0,\ \ \varphi = \theta = \pi/2$$

with respect to $(\dot\theta, \dot\varphi, \theta, \varphi)$.

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